Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at http://about.jstor.org/participate-jstor/individuals/early-journal-content.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

THE
AMERICAN
MATHEMATICAL MONTHLY.

Entered at the Post-office at Springfield, Missouri, as second-class matter.

VOL. VIII. NOVEMBER, 1901. No. 11.

ON THE HISTORY OF SEVERAL FUNDAMENTAL THEOREMS IN THE THEORY OF GROUPS OF FINITE ORDER.[*]

By DR. G. A. MILLER, Leland Stanford Jr. University, Palo Alto, Cal.

The three most prominent sources of the theory of groups of finite order are: geometric transformations, theory of numbers, and theory of algebraic equations. The group properties of the totality of the rotations through a submultiple of 360°, or through 180° around three concurrent lines, each perpendicular to the plane determined by the other two, must have been observed very early. The prominent place which the five regular polyhedra—the Platonic bodies—occupy in the history of thought make it appear probable that the group properties of the totality of the rotations into themselves of these solids were observed quite early.[†]

The elementary theory of congruences and especially the combinatory laws of the n roots of unity when they are multiplied together exhibit group properties which cannot have escaped the notice of the early students of these questions. As Cayley aptly says (*Philosophical Magazine*, 1854, page 40) "A number of elementary group concepts have been employed by mathematicians for a long time but have not been especially noted on account of their great simplicity." Some of the theorems by Gauss and Schering along this line are of sufficient interest in themselves to attract attention. In particular the theorem by Gauss to the effect that an Abelian group can be resolved in only one way into factor groups whose orders are prime to each other, and the theorem by Schering to the effect that every Abelian group can be resolved in more than one way into factor groups such that the order of each is divisible by the orders of all those which follow it, have a strong claim to being considered the beginning of the theory of abstract groups of finite order.

---

[*] Read before the American Association for the Advancement of Science, August, 1901.

[†] Bravais, *Liouville's Journal*, Vol. 14, 1849, page 167.

The early developments in substitution groups were made in connection with the theory of algebraic equations, starting with the theories of Lagrange and becoming prominent in the development and extension of these theories by Ruffini, Abel, and Galois.* Cauchy seems to have been the first to develop the theory of substitution groups independently of any direct application to the theory of equations, and he was also the first to prove any remarkable theorems in this theory. The earlier theorems were almost self-evident, but this cannot be said of the theorem that a group must involve a substitution of a prime order (p) whenever its order is divisible by p.† This theorem, known in the theory of groups as Cauchy's theorem, is especially interesting since it is independent of the particular notation by means of which a group is represented and hence applies directly to the theory of abstract groups.

In proving this theorem, Cauchy employed another interesting theorem, due to himself, viz., that the symmetric group of degree n contains a subgroup of order p^m, p^m being the highest power of p that divides $n!$ About thirty years later Sylow extended very materially these results due to Cauchy by proving that a group (G) of order g must always contain $1+kp$ subgroups of order p^a, p^a being the highest power of p that divides g, and that all of these subgroups are conjugate under G. This fundamental theorem is known as *Sylow's theorem.* It is sometimes called the Cauchy-Sylow theorem, and was first published in the *Mathematische Annalen*, Vol. 5.

About twenty years later Frobenius published, in the *Berliner Sitzungsberichte*, a very important extension of Sylow's theorem in which it is proved that the number of subgroups of order p^a which are contained in any group is $\equiv 1$ mod p. These subgroups do not necessarily form a single set of conjugates unless p^a is the highest power of p that is contained in the order of the group. Frobenius deserves credit also for a simple proof of Sylow's theorem by means of a more prominent use of the concept of complete sets of conjugates. This concept is prominent in a large number of the publications by Frobenius.

From what precedes it may be observed that a very important part of Sylow's theorem was constructed on French soil. A Norwegian greatly enlarged and beautified this structure. Finally a German enhanced its usefulness by extending it very materially in certain directions. Hence this fundamental theorem stands before us as a truly international structure, which has required more than fifty years for its erection. One of its main contributors (Sylow) has published comparatively little along the line of group theory, while the other two have published extensively along this line.

---

*Cauchy, *Physique Mathematique*, Vol. 3, 1844, page 250.

†Pierpont, *Bulletin of the American Mathematical Society*, Vol. 1, 1895, page 196.

The article in which Sylow made his famous theorem known contains another of not much less importance, viz., A group of order p^{α} contains at least p invariant operators. This furnishes the starting point of a large number of the theorems relating to groups whose order is a power of a prime. On account of its fundamental importance it has been proved in several different ways. The most recent of these proofs is deduced from the known fact that the subgroup which omits a given element of a transitive substitution group of order p^m omits a power of p of its elements.*

Closely related to this theorem is the question in regard to the largest Abelian subgroup in a group of order p^{α}. The first step towards a theorem along this line was the observation that every group of order 16 contains an Abelian subgroup of order 8. A little later it was observed (Comptes Rendus, February, 1896) that every group of order p^4 contains an Abelian subgroup of order p^3. This fact was then proved independently of the list of all the possible groups of order p^4.† Finally, it was proved in the *Messenger of Mathematics*, 1897, that every group of order p^{α} contains an Abelian group of order p^m, whenever $\alpha > \frac{m(m-1)}{2}$

A more special theorem which has a somewhat singular history is the one which states that there are just fifty-one groups of order 32. In 1896 a Frenchman stated in *Comptes Rendus de l' Académie des Sciences* that he had found seventy-five groups of this order and had not yet finished his enumeration. Shortly after this an American stated in the same journal that he also had investigated this problem and that he had proved that there are only fifty-one such groups.

About two years later an Italian published in *Annali di Matematica* his investigations in regard to all the groups of order p^5 and stated therein that both of these results were incorrect inasmuch as the correct number of these groups was just 50. Very shortly after this the said American reaffirmed his former results and called attention to several errors in the enumeration of the Italian. Finally, the latter published a separate investigation of this subject in which he agreed with the American and stated that he considered the determination of all the groups of order 32 settled beyond a doubt.

In tracing the history of a concept or theorem the most difficult part is that in which the concept seems implied, for one is always in danger of reading things into a paper, which the author did not have in mind. This is perhaps especially true of the concepts of isomorphisms and group of isomorphisms. Both of these are implied in the theory of intransitive substitution groups, but it is difficult to say whether any of the earliest workers along this line had them distinctly in mind.

It appears that Hölder and Moore were the first to call explicit attention to the fact that the totality of the simple isomorphisms of a group with itself constitute a group. Hölder remarked that the group of cogredient isomorphisms

---

***American Journal of Mathematics*, Vol. 23, page 173.

†*Quarterly Journal of Mathematics*, Vol. 28, page 233.

is an invariant subgroup of the group of isomorphisms. Recently the group of isomorphisms of a group (G) has been studied from various standpoints. It has been observed that it is the largest subgroup of the holomorph of G, which does not include one of the elements. When G is cyclic it is Abelian, but when G is any non-cyclic Abelian group its group of isomorphisms is non-Abelian. In the latter case its invariant operators are those which transform each of the operators of G into the same power.

The second volume of Weber's Algebra is one of the best and most popular works on the theory of groups of finite order. This may perhaps justify our noting a very singular error which occurs on page 54 of the first edition. The author states at this place that the natural numbers when combined by multiplication furnish the most important example of a commutative group. It is very easy to see that these numbers, combined in the said manner, do not form any group at all. So that what is called the most important example has no existence.

In his second edition Weber recognizes this fact and replaces the given example by another "most important example." It would probably be difficult to prove that numbers combined by multiplication furnish a more important example of Abelian groups than when they are combined by addition. The greatness of this work may perhaps justify the noting of another slight defect, viz., in the treatment of the new subject of commutator groups, the author gives reference to the man who first published the name of these groups but he does not give any reference to the one who first published their properties.

SUPPLEMENTARY REPORT ON NON-EUCLIDEAN GEOMETRY.

By DR. GEORGE BRUCE HALSTED, University of Texas, Austin, Texas.

When at the Columbus Meeting of the American Association I had the honor of making a Report on Non-Euclidean Geometry, it was mentioned that my own Bibliography of Hyper-space and Non-Euclidean Geometry in the *American Journal of Mathematics* (1878) giving 81 authors and 174 titles, when reprinted in the collected works of Lobachevski (Kazan, 1886) gives 124 authors and 272 titles ; while Roberto Bonola had just given (1899) a Bibliography of the Foundations of Geometry in relation to Non-Euclidean Geometry containing over 350 titles with some repetitions.

Bonola in 1900 finished a second part of this Bibliography in which the single section headed "Historical, Critical, and Philosophical Writings" gives 96 authors and 150 titles. It thus becomes very evident that a most important function of your Reporter is the selection of what writings to bring forward for especial mention as of paramount importance and typical of the main stream of advance. In the Columbus Report I particularly stressed the work of two